Final Practice

May 1, 2015

Exercise 1

The functions $x(t) = e^{-2t}$ and $y(t) = -e^{-2t}$ are a solution to the system of differential equations

$$\frac{dx}{dt} = 2x + 4y$$
$$\frac{dy}{dt} = -x - 3y.$$

(a) TRUE
(b) FALSE

Exercise 1

The functions $x(t) = e^{-2t}$ and $y(t) = -e^{-2t}$ are a solution to the system of differential equations

$$\frac{dx}{dt} = 2x + 4y$$
$$\frac{dy}{dt} = -x - 3y.$$

(a) TRUE

(b) FALSE

Plug in and check.

Exercise 2

The functions $x(t) = e^{-2t}$ and $y(t) = e^{t}$ are a solution to the system of differential equations

$$\frac{dx}{dt} = 2x + 4y$$
$$\frac{dy}{dt} = -x - 3y.$$

(a) TRUE

(b) FALSE

Exercise 2

The functions $x(t) = e^{-2t}$ and $y(t) = e^t$ are a solution to the system of differential equations

$$\frac{dx}{dt} = 2x + 4y$$
$$\frac{dy}{dt} = -x - 3y.$$

(a) TRUE
(b) FALSE

Plug in and check.

Exercise 3

Find the general solution to the system of differential equations

$$\frac{dx}{dt} = 2x + 4y$$

$$\frac{dy}{dt} = -x - 3y.$$

Write your answer as $x(t) = \ldots$ and $y(t) = \ldots$.

Exercise 3

In all of the below, c_1, c_2 can be ANY real constants.

(a) $x(t) = c_1 e^{-2t} + 4c_2 e^{t}, \quad y(t) = -c_1 e^{-2t} - c_2 e^{t}$

(b) $x(t) = 2c_1 e^{-2t} + 4c_2 e^{t}, \quad y(t) = -2c_1 e^{-2t} - c_2 e^{t}$

(c) $x(t) = c_1 e^{-2t} - c_2 e^{t}, \quad y(t) = -c_1 e^{-2t} + \frac{1}{4} c_2 e^{t}$

(d) $x(t) = c_1 e^{-2t} + 2c_2 e^{t}, \quad y(t) = -c_1 e^{-2t} - \frac{1}{2} c_2 e^{t}$

(e) $x(t) = -c_1 e^{-2t} + 4c_2 e^{t}, \quad y(t) = c_1 e^{-2t} - c_2 e^{t}$

Exercise 3

ALL answers were correct. Those are all correct forms of the general solution, as c_1, c_2 range through all the real numbers those sets of solutions are the same.

Exercise 3, solution, page 1

We first write this system of equations in matrix form:

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

We then have to find the eigenvalues and eigenvectors of the 2×2 coefficient matrix . We have shown that the eigenvalues are those real numbers λ for which

$$\det \begin{pmatrix} 2-\lambda & 4 \\ -1 & -3-\lambda \end{pmatrix} = (2-\lambda)(-3-\lambda) + 4 = \lambda^2 + \lambda - 2$$

is equal to zero. This determinant is equal to

$$(\lambda + 2)(\lambda - 1) = 0$$

and so the eigenvalues are $\lambda = -2$ and $\lambda = 1$.

Exercise 3, solution continued, page 2

We now find the eigenvectors. To find an eigenvector for $\lambda = -2$, we have to solve

$$\begin{pmatrix} 2-(-2) & 4 \\ -1 & -3-(-2) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This comes out as:

$$4u + 4v = 0; \quad -u - v = 0.$$

We can choose any solution that is not u, v both zero, so for example, we can take

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This is therefore an eigenvector with eigenvalue -2.

Exercise 3, solution continued, page 3

To find an eigenvector for $\lambda = 1$, we have to solve

$$\begin{pmatrix} 2-1 & 4 \\ -1 & -3-1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which comes out as

$$u + 4v = 0; \quad -u - 4v = 0$$

so a non-zero solution is

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}.$$

Exercise 3, solution continued, page 4

Putting it all together tells us that the general solution to the original system of differential equations is therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

for c_1, c_2 ANY real numbers,
or

$$\boxed{x(t) = c_1 e^{-2t} + 4c_2 e^{t}, \quad y(t) = -c_1 e^{-2t} - c_2 e^{t}}$$

for c_1, c_2 ANY real numbers.

Exercise 4

$$f(x, y) = \sqrt{4 - x^2 - y^2}.$$

Find the domain and range of f. Sketch the c-level curves of f for $c = -5, 0, 1, 2$

Exercise 4

The domain of $f(x, y) = \sqrt{4 - x^2 - y^2}$ is

(a) $\mathbb{R}^2$

(b) $\{(x, y) \mid x^2 + y^2 \leq 4\}$

(c) $\{(x, y) \mid x^2 + y^2 \geq 4\}$

(d) $\mathbb{R}^2 \backslash \{(0, 0)\}$

(e) The disk of radius 2 around the origin.

Exercise 4

The domain of $f(x, y) = \sqrt{4 - x^2 - y^2}$ is

(a) $\mathbb{R}^2$

(b) $\{(x, y) \mid x^2 + y^2 \leq 4\}$

(c) $\{(x, y) \mid x^2 + y^2 \geq 4\}$

(d) $\mathbb{R}^2 \backslash \{(0, 0)\}$

(e) The disk of radius 2 around the origin.

NOTE: (b) and (e) represent the same subset of $\mathbb{R}^2$. The set

$$\{(x, y) \mid x^2 + y^2 \leq 4\}$$

is the set of points in the disk of radius 2 around the origin.

Exercise 4

The range of $f(x, y) = \sqrt{4 - x^2 - y^2}$ is

(a) $\mathbb{R}$

(b) $\{z \mid z \geq 0\}$

(c) $\{z \mid 0 \leq z \leq 2\}$

(d) $\{z \mid 0 \leq z \leq 4\}$

(e) I don't know.

Exercise 4

The range of $f(x, y) = \sqrt{4 - x^2 - y^2}$ is

(a) $\mathbb{R}$

(b) $\{z \mid z \geq 0\}$

(c) $\{z \mid 0 \leq z \leq 2\}$

(d) $\{z \mid 0 \leq z \leq 4\}$

(e) I don't know.

NOTE

$$0 \leq 4 - x^2 - y^2 \leq 4$$

for (x, y) in the domain of f.

Exercise 4, continued

There are is level curve for $c = -5$ since -5 is not in the range of f.
For any $0 \leq c \leq 2$, we get that the level curve is a circle of radius $\sqrt{4 - c^2}$. In particular, for $c = 0$, we get a circle of radius 2, for $c = 1$, we get a circle of radius $\sqrt{3}$ and for $c = 2$, we get a circle of radius 0, namely just a point.

Exercise 5

The function $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ is continuous everywhere on its domain.

(a) TRUE

(b) FALSE

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The function $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ is continuous everywhere on its domain.

(a) TRUE

(b) FALSE

Note that the domain of f excludes the point $(x, y) = (0, 0)$, so it would not even make sense to ask for the function to be continuous there. We can ask for the limit at a point on that line, but we can't compare that with the value of the function there since the function is not defined there.

Exercise 6, part 1

Consider the function

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}.$$

Compute the limit

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

along the x-axis and along the y-axis. What can you conclude about the existence of the limit?

Exercise 6, part 1

(a) The limit exists.
(b) The limit does not exist.
(c) I didn't conclude anything.

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(b) The limit does not exist.

(c) I didn't conclude anything.

Exercise 6, part 1

For $y = 0$, $f(x, y) = \dfrac{x^2}{x^2} = 1$, so along the x-axis, $\lim_{(x,y)\to(0,0)} = 1$.
For $x = 0$, $f(x, y) = \dfrac{-y^2}{y^2} = -1$, so along the y-axis, $\lim_{(x,y)\to(0,0)} = -1$.
Since we got two different limits along two different paths, we can conclude that the limit does not exist.

Exercise 6, part 2

The function

$$g(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & if (x,y) \neq (0,0) \\ 0 & if (x,y) = (0,0) \end{cases}$$

is continuous everywhere on its domain.

(a) TRUE

(b) FALSE

Exercise 6, part 2

The function

$$g(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & if (x,y) \neq (0,0) \\ 0 & if (x,y) = (0,0) \end{cases}$$

is continuous everywhere on its domain.

(a) TRUE

(b) FALSE

This function is clearly continuous everywhere except possibly at (0,0), and it is now defined at (0,0), so we can ask about continuity there. Recall that the function would be continuous at that point if

$$\lim_{(x,y)\to(0,0)} g(x,y) = g(0,0) = 0.$$

But we already computed the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

and showed that it doesn't exist, so it's not equal to 0.

Exercise 7, part 1

Consider the function

$$f(x, y) = \frac{xy}{x^2 + y^2}.$$

Compute the limit

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

along the x-axis and along the y-axis. What can you conclude about the existence of the limit?

Exercise 7, part 1

(a) The limit exists.

(b) The limit does not exist.

(c) I didn't conclude anything.

Exercise 7, part 1

If $y = 0$, $f(x, 0) = \dfrac{0}{x^2} = 0$, so along the x-axis, $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0$.
If $x = 0$, $f(0, y) = \dfrac{0}{y^2} = 0$, so along the y-axis, $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0$.
BUT, the fact that the limits agree on some paths does not mean that the limit exists. RECALL:

- in order to show a limit does not exist, it's enough to show the z-values approach different numbers on 2 different paths.
- in order to show a limit exists you would have to show it using some theorem about existence of limits such as the properties of limits or, if those don't apply, the squeeze theorem.

Exercise 7, part 2

Still about the function

$$f(x, y) = \frac{xy}{x^2 + y^2}.$$

Compute the limit

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

along the $x = y$ line. What can you conclude about the existence of the limit?

Exercise 6, part 2

(a) The limit exists.
(b) The limit does not exist.
(c) I didn't conclude anything.

Exercise 6, part 2

(a) The limit exists.

(b) The limit does not exist.

(c) I didn't conclude anything.

Exercise 6, part 2

For all $x \neq 0$, along $x = y$,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,x)\to(0,0)} \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

Since this limit is not the same with the one we obtained going along the axes, we can conclude that the limit does not exist.